

MARCHETTI-062

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A Simple Mathematical Model of War Events

A methodology is presented capable of modeling social affairs with great mathematical parsimony. The basic hypothesis is that human actions are the physical consequence of the cultural diffusion of action models or action paradigms. Sequences of events can then be modeled using *diffusion equations* which, in the simplest cases, are *logistics*.

To illustrate the applicability of this concept to historical events we have analyzed a set of data relating to World War I, World War II, and the Vietnam War. The analysis will reveal a property we observed for social behavior in general, i.e., the system is (quasi) *fractal*, meaning that the same equations (but with different parameters) can describe it at various hierarchical levels.

As J.A. Hobson said in 1894, « To understand the growth of the industrial organism we must apply to the heaps of bare unordered facts those principles of order derivable from the other sciences and those general laws of development which seem common to all bodies of closely related phenomena ». For almost 20 years we have applied this guideline to an increasingly variegated set of examples of social behavior, now amounting to about 3,000 cases analyzed, using a conceptually very simple methodology which reduces behavior to the cultural diffusion of *behavioral examples or paradigms*.

In the simplest case these paradigms diffuse through *personal contacts*, at all levels of social interaction, with formal characteristics analogous to an epidemic. In fact, the mathematics we have adopted is the same as that of an epidemic, which again in the simplest case reduces to a three-parameter logistic (see a brief explanation in the Appendix).

In this paper we explore the possibility of extending the application of the model to a type of event of great interest to historians : *war*. War can be an extension of diplomacy by other means, but the new means represents a discontinuity in the process, i.e., there must be a buildup of forces and of their activity which represent a *reorganization of the system presumably spreading in epidemic fashion*. Numerous examples from the economic realm show this type of behavior and it is very natural to extend the concept to a war situation.

Putting data relating to war efforts into equations has two purposes of interest for the historian. One is to quantify long-term processes by

means of simple mathematical frames to facilitate the search for rules and taxonomic relations. The second is to look for invariants that may lead to predictions. Logistic equation modeling in fact has shown a robust predictive capacity in many fields and by analogy good results were expected also in this case. After all, human behavior appears very homogeneous if we are capable of finding a model abstracting from the triviality of accidents.

This modeling being of quantitative character, we have to look for numerical indicators that describe the events. The best recorded indicator of the level of friction between two armies is the number of deaths. Death is a serious thing even in war, the records kept are reasonably accurate and there is not much ambiguity on what to count. So we will start by modeling, with an interpolating logistic, the number of deaths in the British Army during World War I. The result of the logistic fitting in the period 1915-1919 is reported in figure 1. The points represent the *cumulative* number of deaths, and the interpolating line is an equation of the form :

$$N(t) = N / [1 + \exp - (at + b)].$$

The three parameters N (final cumulative number of deaths), a (a number indicative of the rate of the process, i.e., of its distribution in time), and b (a number positioning the event in calendar time), are calculated by iteration fitting. The logistic, which is an S-curve, appears here as a straight line because of a mathematical transform we use to simplify the graphics (see Appendix). More comments about the charts are reported in the extended legends.

The fitting appears very good except for the final swirl, to which we will return after examining other cases of terminal flares. The center-point of the effort appears to be May 1917. On the same figure the British Army strength (in millions) is also reported. It reached its centerpoint in November 1915. The points to make the fit refer to the same month of the year, because wars, for strategic, tactical, cultural, and meteorological reasons, have a strong seasonal component. When we consider the *fine details of the yearly operations*, their structure is perhaps even more remarkable, because each year can be described by its own logistics, roughly encompassing the calendar year and centered in August (fig. 2). The sum of the saturation levels does not match exactly that of the envelope logistic of figure 2 as they are « virtual » saturation points coming from the fitting procedure, and the process does not necessarily go to 100 % saturation. The differences, however, are small.

The system is like a set of Chinese boxes where the mechanism of the small one is homologous to that of the larger one that contains it : it is possible to proceed to analyze lower hierarchical levels, e.g., single battles. This formal identity of mechanisms at different hierarchical levels is a feature we observed in all socioeconomic systems we analyzed that way, and it is presumably related to the fact that diffusing and processing information always follows identical mechanisms. This

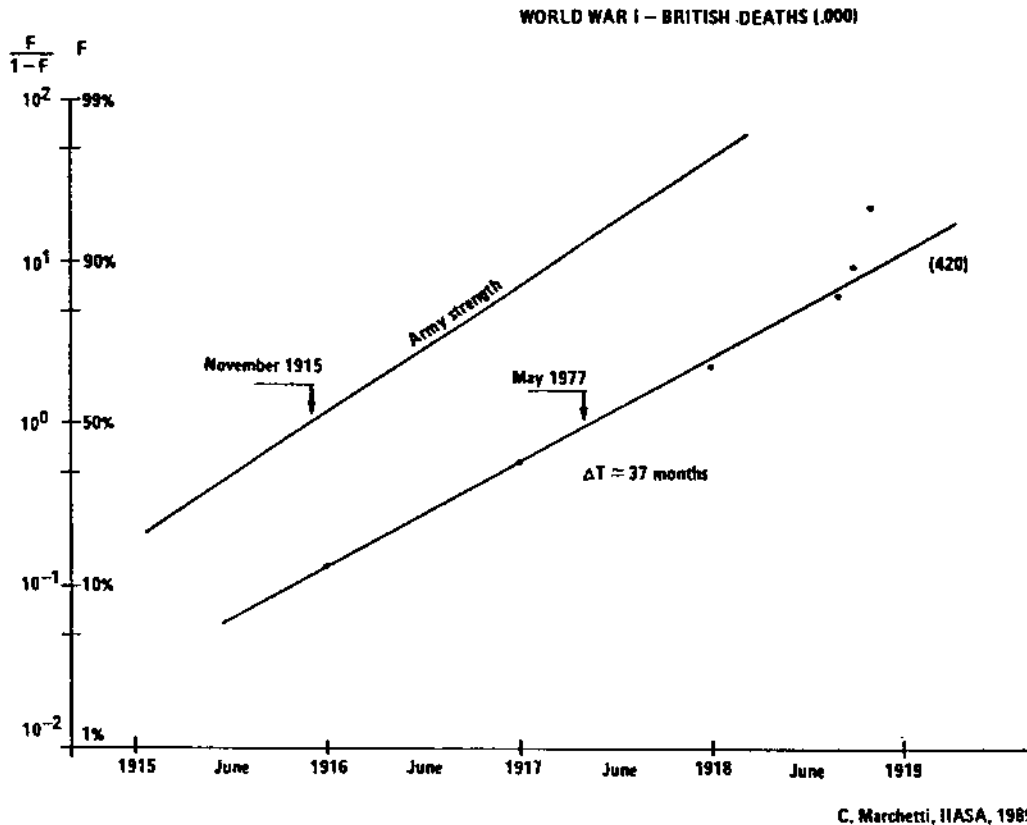


Figure 1: World War I — British Army strength and British casualties (cumulative number, in thousands).

In this chart two logistic analyses are reported together. In the upper line the evolution of the British Army strength is shown. The virtual saturation point, calculated by logistic interpolation of the data (see Appendix for details), is 1.84 million men: this was almost reached in practice. The central point, when the strength is 50 % of the saturation level, was in November 1915. The buildup time (the time to go from 10 % to 90 % of full strength) for the Army's strength is 32 months. In the lower curve we examine the evolution of British casualties (cumulative). Again, the calculated saturation point of 420,000 was almost reached. The center point is in May 1917 (110,000 deaths) and the time constant (from 10 % to 90 % of the saturation level) is 37 months. As explained in the Appendix, the fitting equations are S-curves, «straightened» here for easier presentation. It is remarkable how such an extremely simple model fits so well *the whole of* the evolution of the army's strength and casualties when war appears so contingent and so reliant on the interaction between two *opposing decisional systems*. (Data from His Majesty's Stationery Office (1922).)

shows our society has a neat *fractal* (more precisely quasi-fractal) *structure*, a feature we think is of further interest for the historian as well as the sociologist.

Looking at the situation of the Germans facing the British we find an almost perfect symmetry for the time distribution of the indicators of friction casualties, with *exactly* the same centerpoint in May 1917 and

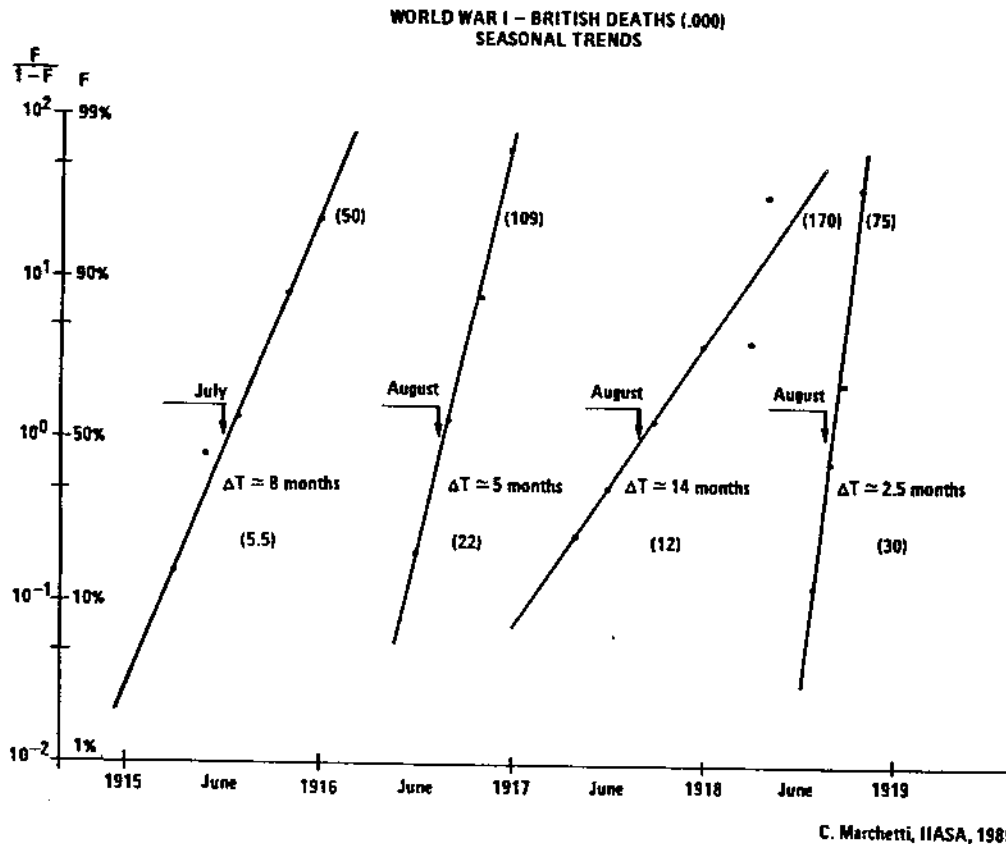
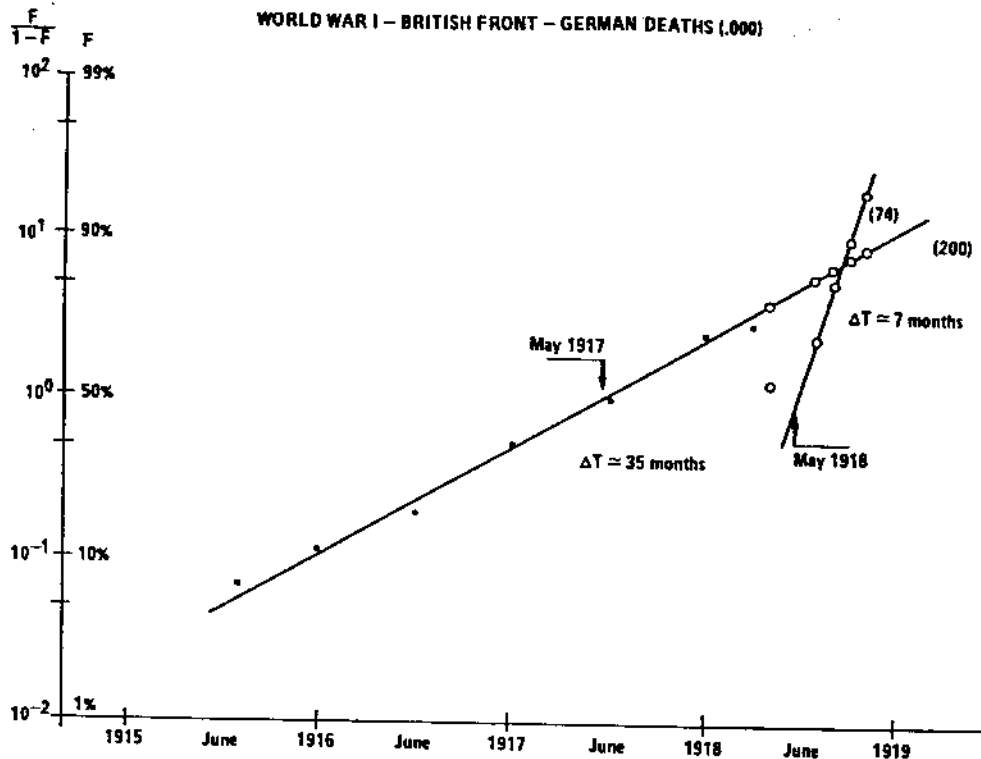


Figure 2 : World War I — British casualties seasonal analysis (in thousands). The analysis of the casualties shown in figure 1 has been done using annual data. As stated in the text, the system is fractal, which means that the same equation can describe it at any level of aggregation or disaggregation. War was always a seasonal affair in ancient times and the middle ages, but even now, in spite of « all-weather weapons », it still maintains a seasonal imprint. We checked this assumption by logistically interpolating (cumulative) monthly casualties. We found a self-consistent pattern *within each year*. The calculated saturation points are quite different from year to year (in thousands, 50, 109, 170, 75) but the central point, at which the *rate of casualties* is at its *maximum value*, is always *August* (with one in July). The sum of the saturation points is slightly lower than the saturation point of figure 1 due to computational errors in the interpolations and the fact that processes are not necessarily completed.

very similar time constant; (fig. 3). The following features should be observed. One is the consistently lower level of casualties on the German side, roughly *one half of the British casualty level*. The second feature is the last final desperate *spurt* killing 74,000 soldiers, which has only a corresponding « swirl » on the British side. That *spurt* did leave intact the previous, long-term pulse. It can be interpreted as a blow to steal the victory; however, it did not change the final response of the war. Incidentally, the 90% point on the casualties curve is reached both for British and for Germans in May 1917 + (37 months/2), i.e., in November 1918.



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Figure 3 : World War I — German casualties at the British front (cumulative number, in thousands).

This is the same analysis as figure 1, but for German casualties. As one might expect, there is a mirror symmetry in the two events. The central point is exactly the same, May 1917, and the time constant is not very different, 37 months compared with 35 months. The striking difference is in the final spurt of activity, i.e., a second wave of casualties superposing on the first one. It is very short (having a time constant of 7 months) and centered in May 1918. This wave, with 74,000 casualties, is very large if we compare it with the « slight hiccup » at the end of the British casualty curve. It is also large in terms of the final total calculated level of 200,000 casualties in the long-term pulse. Incidentally, war terminated when about 90% of this long-term wave was reached toward the end of 1918 (May 1917 + (35 months)/2) November 1918). One may postulate that this last pulse of activity of Germans on the attack was useless in terms of the general outcome of the war. Another remarkable point is that the level of German casualties remained consistently at about 50 % of the British level during the long-term pulse.

Because normally a war ends when one of the two competitors *feels* he lost, one may *speculate* that this feeling becomes action when 90 % of the process is already consumed. We will look at this empirical observation again when examining World War II and the Vietnam War.

Let us now stay with World War I and look at the smaller boxes from a *fractal analysis* point of view. A book by Ayres (1919) reports on many details of the American war effort in Europe : we will concentrate on their analysis. Let us consider first the US Army strength in France : see figure 4. We see here a remarkable speed in the buildup of an army

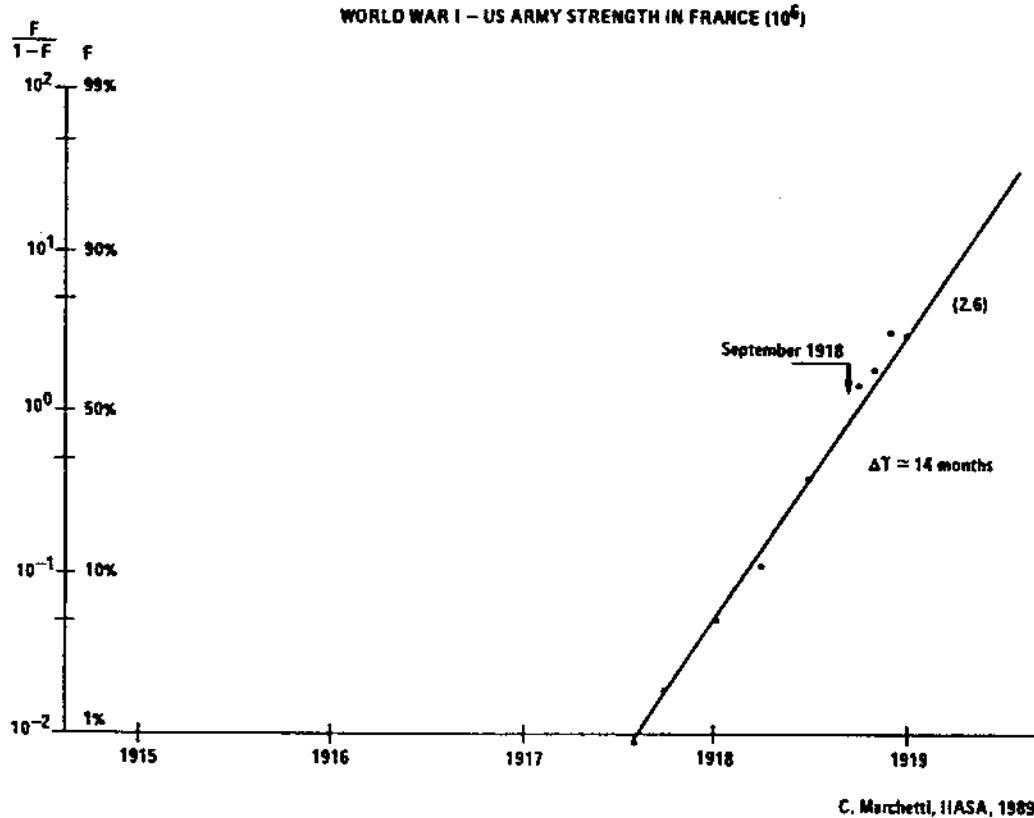


Figure 4 : World War I — US Army strength in France (in millions). The USA did intervene late in the European conflict and the first point in the chart (25,000 men) is in the second half of 1917. But the buildup was very fast, the time constant being 14 months, which may be compared with 32 months for the British Army. The saturation level of 2.6 million was never reached because peace came a few months after the middle point of the buildup in September 1918. (Data from Ayres (1919).)

with a « virtual » saturation point of 2.6 million men with a time constant of only 14 months, half the time constant (Delta) of the British Army. The actual final number corresponds to about three quarters of the saturation point because the end of the war stopped the process of growth. The central point (50 % saturation) is in fact reached only in September 1918 (November 1915 for the British Army).

Focusing on further details, we can analyze the delivery of artillery units to US forces in Europe (fig. 5). We again get an excellent fit with a logistic equation with a virtual saturation point of 3,500 units. In fact, only 90 % of that value was reached. The process of delivery did continue after the end of the war and the central point is slightly late (two months) with respect to the buildup of troops. Also the time constant is slightly shorter (12 months compared with 14 months). Artillery needs the appropriate armament : the delivery of artillery rounds is reported in figure 6. Again, the matching between the data and the model is remarkable. A more detailed discussion is reported in the extended legend.

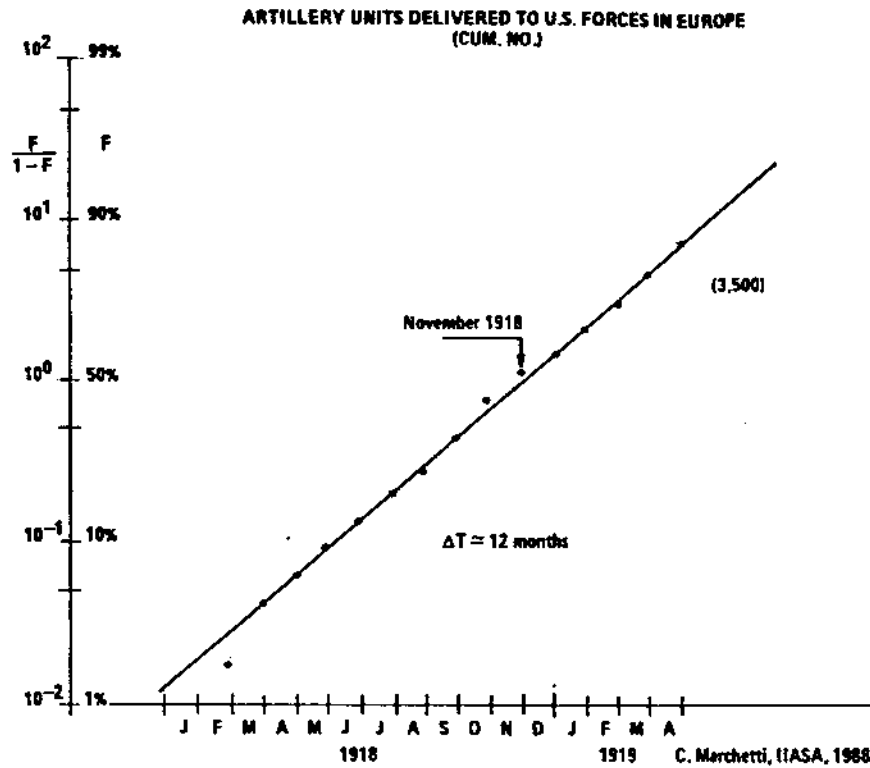


Figure 5: World War I — Artillery units delivered to US forces in Europe (cumulative number, in thousands).
 The analysis of the delivery of artillery units is carried out here month by month for 1918 and part of 1919. Presumably some units were in the pipeline and delivery of these continued after the end of the war. The central point, when deliveries grew at the maximum rate, is located in November 1918, right at the end of the war (November 11). In spite of the fine time division, deliveries fit the logistic diffusion paradigm almost perfectly. (Data from Ayres (1919).)

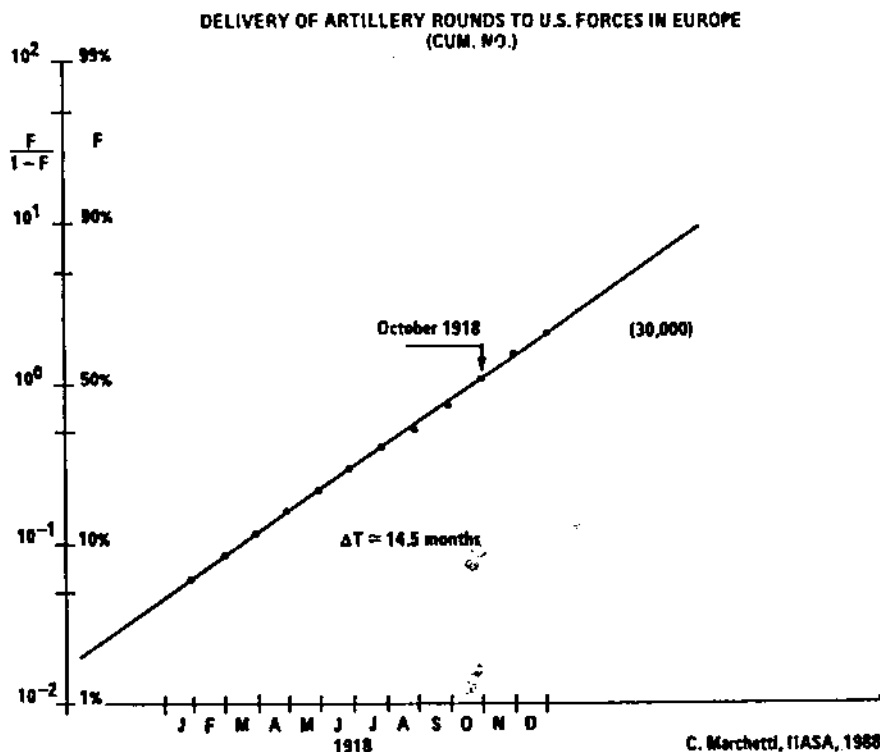


Figure 6: World War I — Artillery rounds delivered to US forces in Europe (cumulative number, in thousands).
 Analysis of ammunition deliveries again shows the usual pattern with the usual regularity around the logistic interpolating equation. Of the 30,000 rounds calculated at saturation point, a little more than half were actually delivered due to the end of the war. The middle point is in fact in October 1918. The time constant and middle point almost coincide with those of the US forces deployment in Europe (fig. 4). (Data from Ayres (1919).)

As regards light armament, the delivery of machine guns has been analyzed in figure 7, that of toxic gases in figure 8. Looking at the air force, the evolution of the air fleet stationed in Europe is reported in figure 9. The maximum buildup rate was in October 1918, when the number of planes on European soil was already 6,000. It stopped growing a couple of months later due to the end of the war. The virtual saturation point is 1,200, this meaning that had the war continued the number of planes delivered would have approached this figure.

The number of planes on the front evolved in time as indicated in figure 10. In July 1918 when 1,400 planes were at the front, their number corresponded more or less to total deliveries up to that date. In October 1918, 6,000 were delivered but less than one third were on the front showing a certain viscosity in making them operational. The logistic paradigm shows again a crisp matching with the dynamics of things in the real world.

Let us now come to World War II. The buildup of US Army manpower is reported in figure 11. The calculated saturation point is seven million, a number never reached because of the end of the war. In

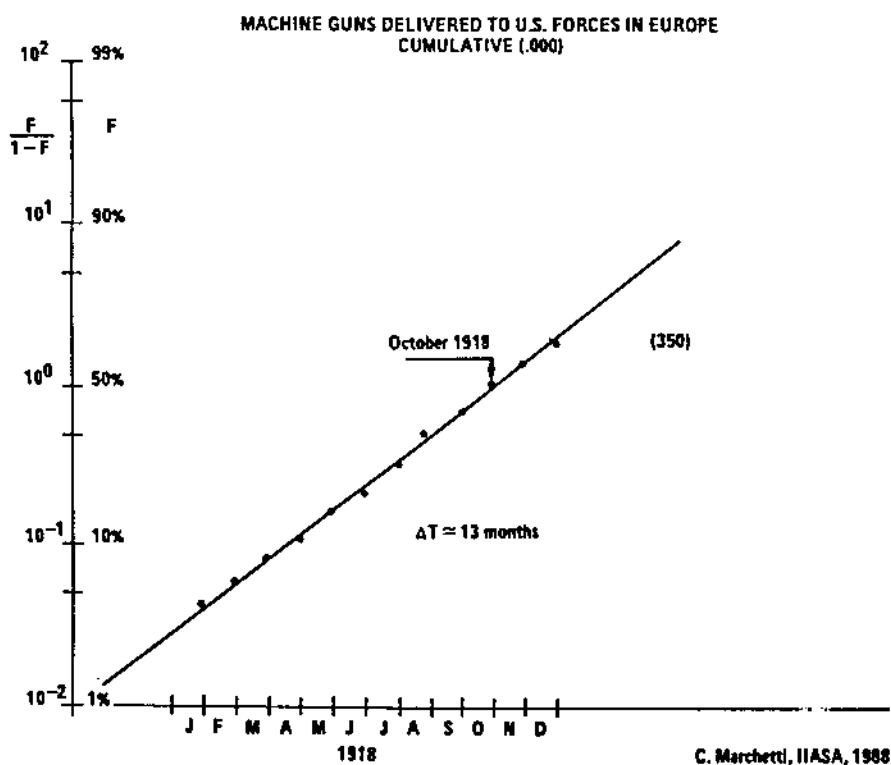


Figure 7: World War I — Machine guns delivered to US forces in Europe (cumulative number, in thousands).

Looking at the delivery of light weapons, the case of machine guns is analyzed here. The taxonomic characteristics of their deployment are almost identical to those of artillery and artillery rounds, showing that these deliveries are part of the same plan. (Data from Ayres (1919).)

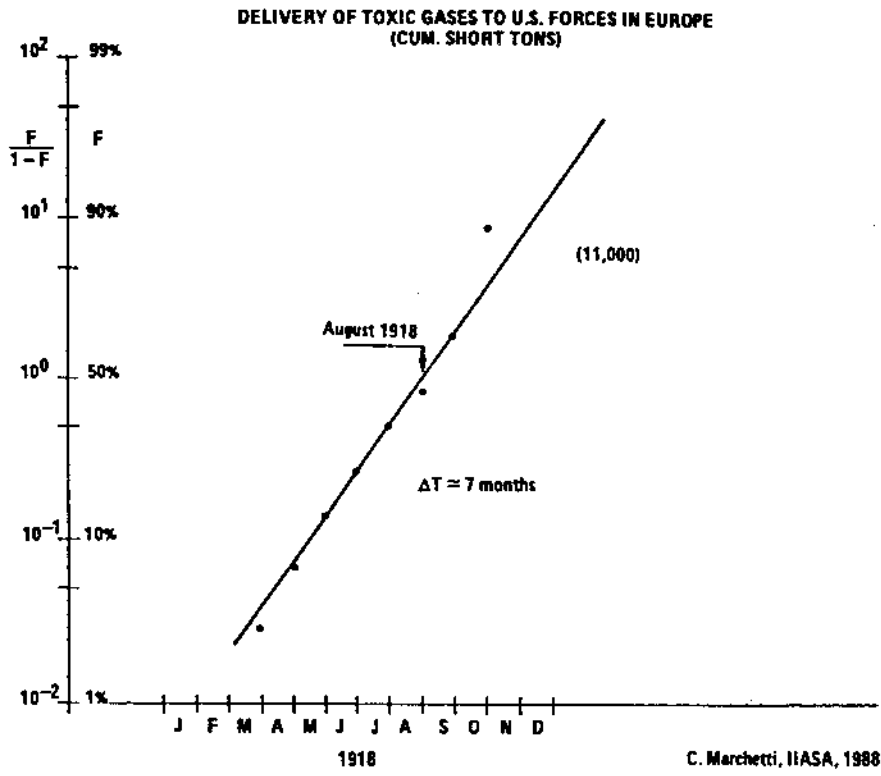


Figure 8: World War I — Toxic gases delivered to US forces in Europe (cumulative short tons). Short tons of toxic gas deliveries (cumulative) to US forces in Europe are modeled here. Taxonomically they are a little earlier than, for example, artillery rounds (center point August 1918 compared with October 1918) are more concentrated in time (time constant 7 months compared with 14.5 months). (Data from Ayres 1919.)

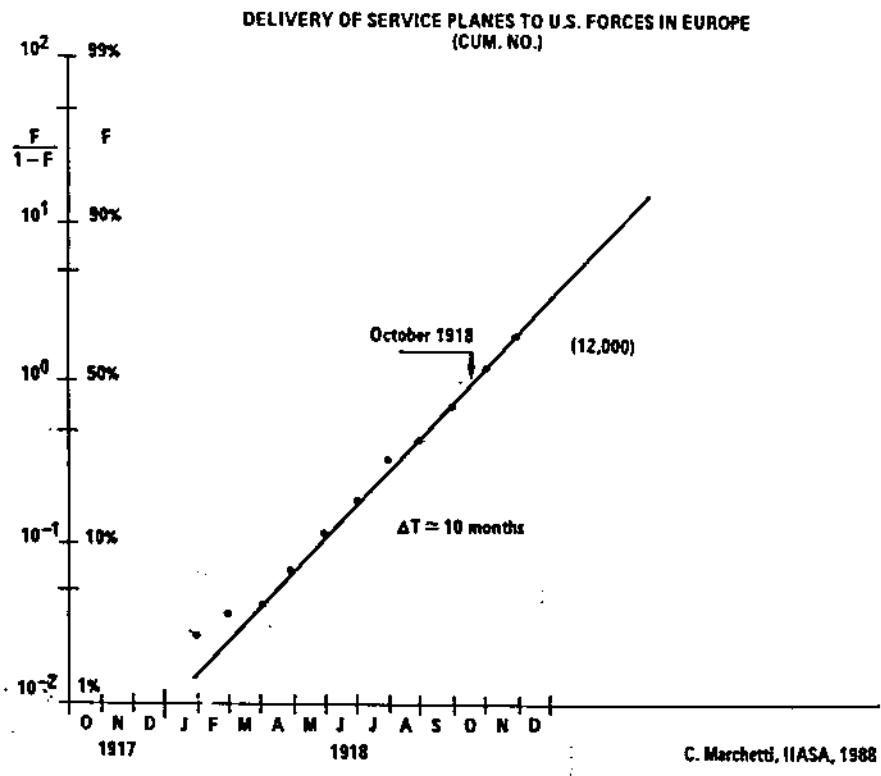


Figure 9: World War I — Airplanes deliveries to US forces in Europe (cumulative number). Airplanes were an exotic weapon in World War I and an obviously more complex machine than shells and guns. One might have expected the taxonomy of the delivery process to have special characteristics. However, with a center point in October 1918 and a time constant of 10 months airplanes are in much the same range as artillery units. (Data from Ayres (1919).)

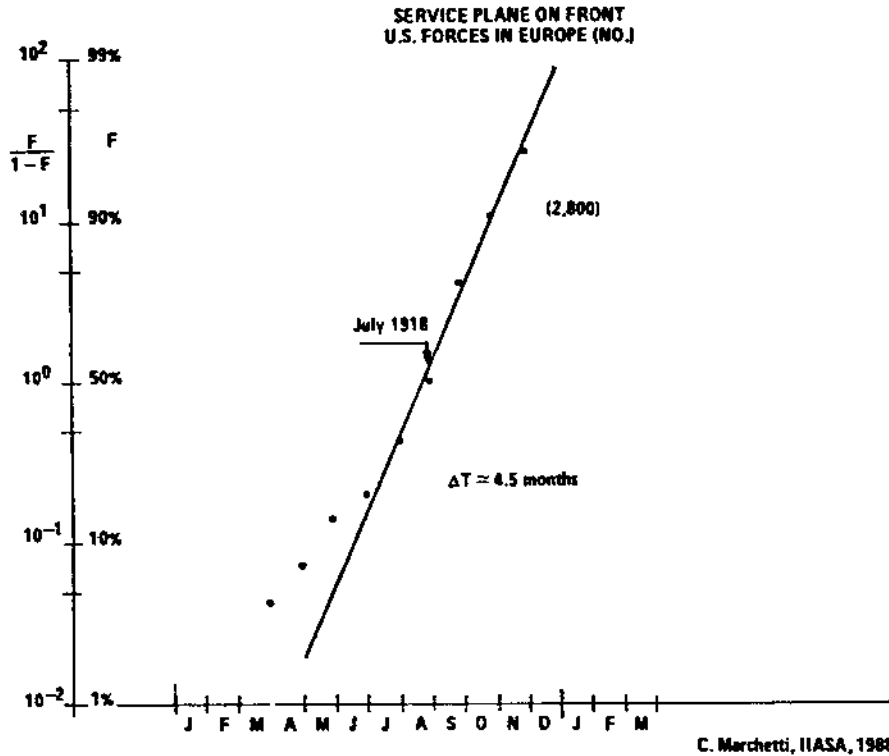


Figure 10: World War I — Airplanes at the front — US forces in Europe (actual number).

Compared with the analysis of figure 9, it appears that all planes available until July or August 1918 were sent to the front line. Most of the late arrivals were presumably kept as backups. It is interesting to note that the dynamics of the presence of planes at the front is self-consistent and, to a point, independent of the dynamics of deliveries. (Data from Ayres (1919).)

World War I also the Americans tended to be latecomers and were caught by the end of the war in the middle of their buildup of military strength. If we come to casualties, however, (fig. 12) we have a completely different picture. Here we have two pulses, one due to the war in general and the second presumably linked to the final attack in Europe. The war ends when *both of them* reach about 90 % of the *calculated* saturation value. The same happened for British *and* German casualties in World War I and for American casualties in the Vietnam War.

The analysis of American casualties in the Vietnam War (fig. 13) shows two clearly separable impulses. This leads to the suggestion, from a purely taxonomic consideration, of looking at the actual description of the war to see if this separation corresponds to a real split in operational or political terms.

As stated at various points in this paper, our world is fractal, in the sense that taxonomies are homologous at various aggregations of hierarchical levels. We conclude by showing how, by counting conflicts

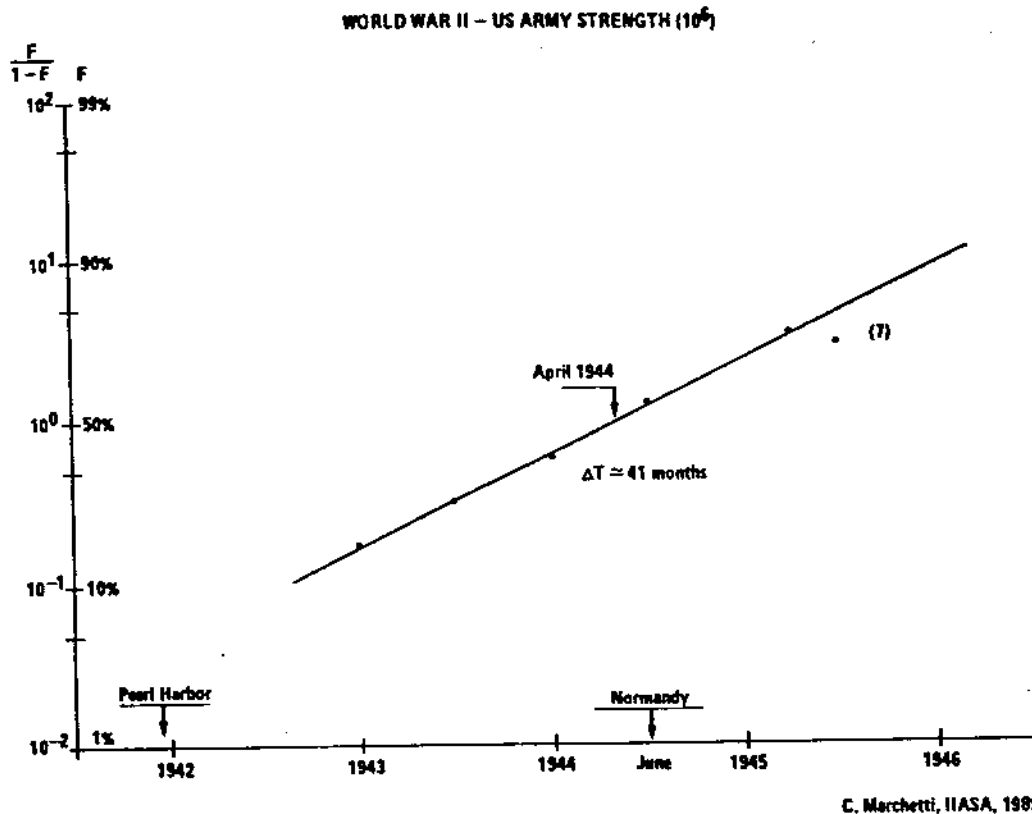


Figure 11 : World War II — Buildup of US Army (millions).
 The buildup up of a large army from scratch is a very complex operation, but the actual time taken is relatively short. In the case of the British Army of World War I (fig. 1), the time constant to set up an army with (virtual) 1.8 million men was 32 months. For the Americans in World War II, with a (virtual) 7 million men at saturation, the time constant was 41 months. As in World War I, Americans were latecomers. The central point, where there was maximum growth, was in April 1944, just a year before the end of the war in Europe. This is an improvement with respect to World War I where the maximum rate of growth was in September 1918, two months before the armistice. The end of the war stopped the buildup : the virtual saturation point was 7 million, as stated. (Data from Annual Report from the Secretary of the Army 1948 (1949).)

at world level, using a self-consistent definition and cutoff, one gets a time series that fits the paradigm (fig. 14). The message portrayed by the analysis is consoling in the sense that conflictuality seems to be fading out. However, the size of the conflicts tends to grow as a compensating factor for their lower number.

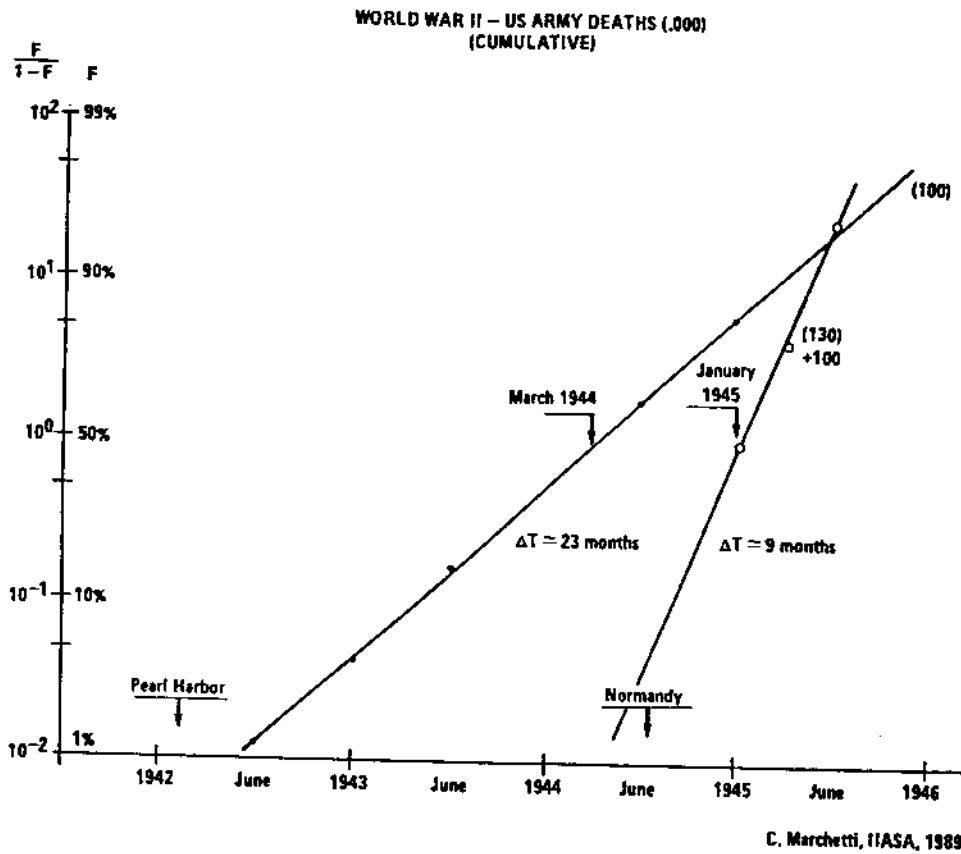


Figure 12: World War II — US Army casualties (cumulative number, in thousands).
 Analysis of US Army casualties shows a different pattern than that of figure 11. The US casualties can be sub-divided into two pulses, one centered in March 1944 with 100,000 (virtual) deaths and a time constant of 23 months, the other centered in January 1945 with 130,000 (virtual) deaths and a time constant of 9 months. The second pulse may be attributed to the Normandy operation as the temporal coincidence suggests, although a recrudescence of activity on all fronts might have contributed to it. We find it very interesting that the war ended when 90 % of the calculated virtual casualties had occurred, as in the case of the British Army during World War I (fig. 1) or the Germans in World War I (fig. 3). Because the saturation point for the fitting equation can be calculated at a relatively early point in time, we *might* have here a tool to estimate when a certain conflict will finally end. (Data from Annual Report from the Secretary of the Army 1948 (1949).)

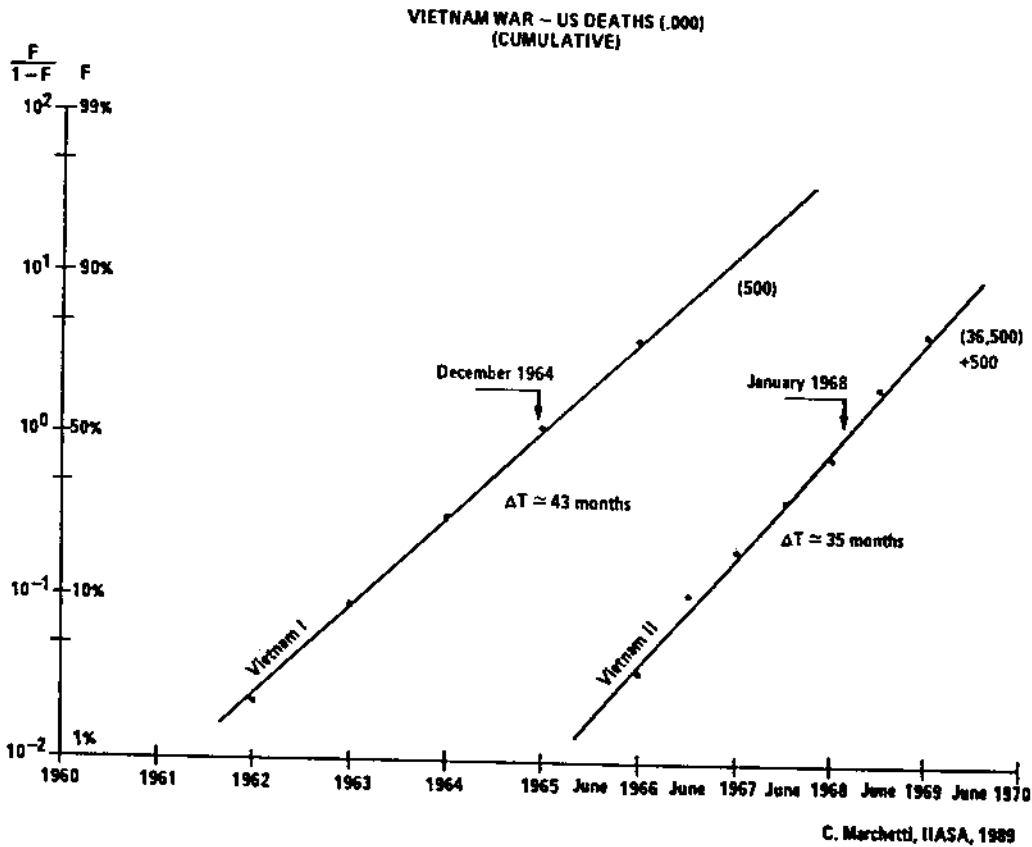


Figure 13 : Vietnam War — US casualties (cumulative number).
 The analysis of the number of casualties is made independently from any other contextual information and shows that the Vietnam War was in fact a sequence of 2 wars of similar length (time constants 43 months and 35 months, respectively), placed about 3 years apart, but with a very different number of virtual casualties (500 and 36,500). In spite of the vastly different number of casualties, each war ended when about 90 % of the saturation level was reached. We again highlight this coincidence as a suggestion for a possible mechanism to estimate when a war will end. (Data from Office of Assistant Secretary of Defense, US.)

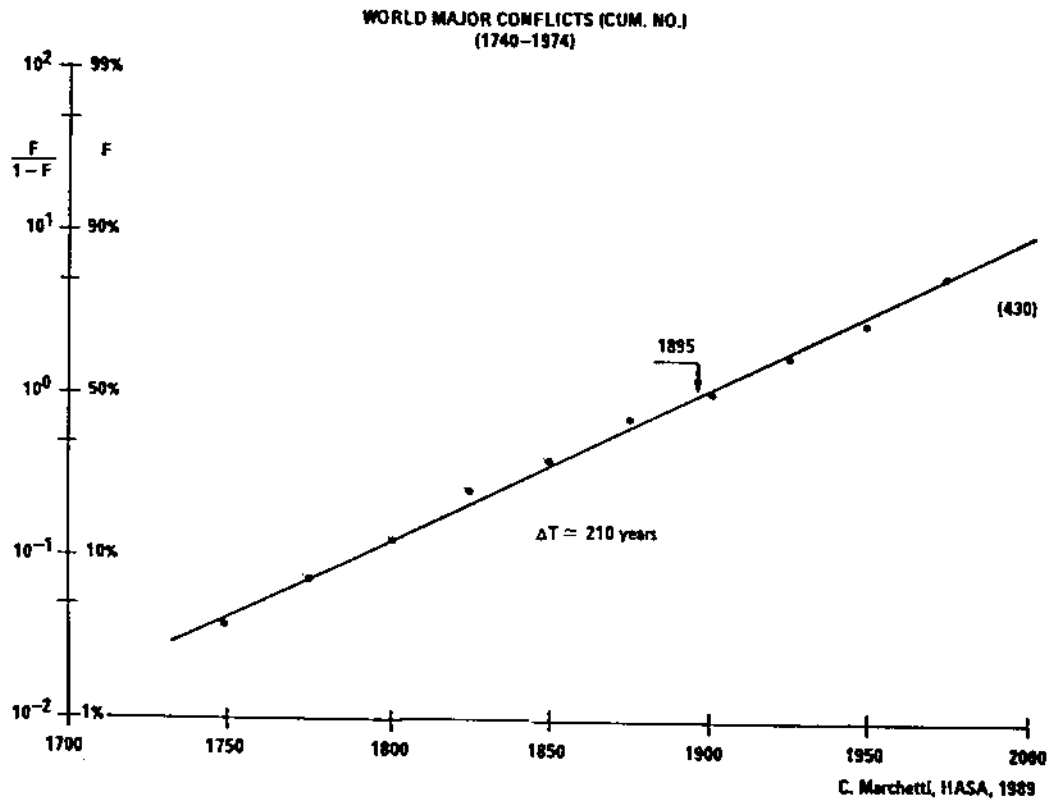


Figure 14: World density of conflicts.

As stated in the text, the analysis of social and economic systems using the methodology proposed in this paper shows that the taxonomy of the social system is fractal. This means that, from the point of view of mathematical description, the hierarchical level and the level of aggregation are immaterial. As we have analyzed wars *per se*, i.e., of their density in history from a compilation by Bouthoul and Carrère (1976). The cumulative number of wars in the last two centuries neatly fits a single logistic equation. The central point, when international aggressiveness was at a maximum, is located around 1900. We are now approaching saturation, which is a good omen. The compilation, however, referred to numbers of conflicts above a certain size. The largest conflicts kept increasing in size somehow compensating for their decreasing number.

Appendix : The Mathematical Methodology

The mathematics used in this analysis is extremely simple. Because historians may not be familiar with it, we add this note for illustration. The basic concept that *action paradigms* diffuse epidemically, is condensed in the epidemic equation :

$$[dN = aN (N - N) dt]$$

saying that the number of *new* adopters (dN) during time dt is proportional (a) to the number of actual adopters (N) multiplied by the number of potential adopters ($N - N$), where N is the final number of adopters.

The integration of this equation gives :

$$[N = N/[1 + \exp - (at + b)]]$$

which is the expression of a logistic S-curve well known to epidemiologists and demographers. *We apply it to ideas.*

In the charts of the present paper the logistic equation is presented in an intuitively more pregnant form. N is measured in relative terms as fraction of N ($F = N/N$), and the S-curve is « straightened » by plotting $\log (F/1 - F)$ (Fisher-Pry transform).

The time constant ΔT is the time to go from $F = 0.1$ to $F = 0.9$. It takes the central part of the process (80 %) and the relation between ΔT and the a in the equation is :

$$[\Delta T = 4.39/a]$$

The central date t_0 is defined as :

$$[b/a]$$

The final number of adopters N is given as a number in parenthesis.

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